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## NOTE ON POLES AND POLARS.

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While the matter here presented is not new to the mathematician, the method appears, from a pedagogical point of view, to have some advantages.

Through the point  $O \equiv (x', y')$  a straight line is drawn cutting a conic  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  at  $A$  and  $B$ .

Let  $C$  be a point in  $AB$  such that  $OC = \frac{2OA \times OB}{OA + OB}$ .

We propose to find the locus of  $C$ , when the straight line turns about the point  $O$ .

Transferring the origin to  $(x', y')$  the new equation is

$$ax^2 + 2hxy + by^2 + 2(ax' + hy' + g) + 2(hx' + by' + f) + ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c = 0.$$

For the sake of brevity we write this

$$ax^2 + 2hx'y' + by'^2 + 2g'x' + 2f'y' + c' = 0,$$

where  $g' = ax' + hy' + g$ ,  $f' = hx' + by' + f$ ,  $c' = ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c$  etc.

Putting  $x = r \cos \theta$ ,  $y = r \sin \theta$ , and arranging according to powers of  $r$ ,

$$(a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta) r^2 + 2(g' \cos \theta + f' \sin \theta) r + c' = 0.$$

$$\text{Then } OC = \frac{2 \text{ product of roots}}{\text{sum of roots}} = \frac{-c'}{g'\cos\theta + f'\sin\theta}.$$

Putting  $OC = \rho$ , we have  $g'\rho\cos\theta + f'\rho\sin\theta + c' = 0$ .

But  $\rho\cos\theta = x$ ,  $\rho\sin\theta = y$ ; hence  $g'x + f'y + c' = 0$ .

This straight line is the polar of the point  $(0, 0)$  with respect to the conic  $ax^2 + 2hxy + by^2 + 2g'x + 2f'y + c' = 0$ .

Transferring to the old origin the equation of the polar becomes

$$(ax' + hy' + g)(x - x') + (hx' + by' + f)(y - y') + ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c = 0,$$

which may be written

$$axx' + h(xy' + x'y) + bgy' + g(x + x') + f(y + y') + c = 0.$$

This is the equation of the polar of  $x'y'$  with respect to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

If the point  $C$  be fixed then will  $O$  describe the polar of  $C$ ; for, we have on putting  $OC = r$ ,  $CA = a$ ,  $CB = b$ ,

$$r = \frac{2(r+a)(r+b)}{2r+a+b}.$$

Clearing and solving for  $r$ , we find  $-r = \frac{2ab}{a+b}$ , i. e.  $CO = \frac{2CA \times CB}{CA + CB}$ .

Hence: *If the point  $P$  lies on the polar of the point  $Q$ , then will  $Q$  lie on the polar of  $P$ .*

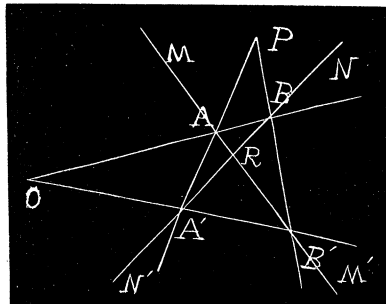
If the line  $OAB$  is drawn tangent to the curve the points  $A$  and  $B$  coincide with  $C$ . Hence:

*The polar of a point passes through the points of contact of the tangents drawn through the point.*

If  $O$  be taken on the curve,  $C$  will coincide with  $O$ , and the polar is the tangent at  $O$ .

The polar of a point with respect to a pair of straight lines passes through the intersection of the lines; for, on drawing a line through  $O$  and the point of intersection,  $A$  and  $B$  coincide with  $C$ .

*To draw the polar of a point with respect to a pair of straight lines:* Through  $O$  draw two secants cutting the lines at  $A$  and  $B$ ,  $A'$  and  $B'$ , respectively. Join  $AA'$ ,  $BB'$  and produce till they meet at  $P$ . Join  $P$  with common point of given lines  $R$ .  $PR$  is the polar. For, the polar of  $AA'$  and  $BB'$  passes through  $P$  and cuts  $AB$  and  $A'B'$  at the same points as the polar with respect to  $NN'$  and  $MM'$ . The line  $OR$  is the polar of  $P$  with respect to either of the two pairs of



lines.  $OP$  is the polar of  $R$  with respect to the same lines ; in other words the triangle  $OPR$  is self-conjugate.

*To draw the polar of a point with respect to a conic:* Draw any two secants through  $O$ , join  $AB'$ ,  $A'B$ ,  $AA'$ ,  $BB'$ . The polar with respect to the conic is the same as the polar with respect to  $AB'$  and  $A'B$ , or with respect to  $AA'$   $BB'$ .

*Construction of Pole and Polar when center is given:* The tangents at points  $(x_1, y_1)$   $(x_2, y_2)$  of the curve are

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0,$$

$$axx_2 + h(xy_2 + x_2y) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0.$$

Subtraction gives

$$ax(x_1 - x_2) + h(xy_1 - y_2) + hy(x_1 - x_2) + by(y_1 - y_2) + g(x_1 - x_2) + f(y_1 - y_2) = 0,$$

$$\text{or } (ax + hy + g)(x_1 - x_2) + (hx + by + f)(y_1 - y_2) = 0,$$

the equation of a line through the intersection of the tangents. This line passes through the center, since the coördinates of the center cause  $ax + hy + g$ ,  $hx + by + f$ , to vanish.

Putting  $y_1 - y_2 = m(x_1 - x_2)$ , we may write this equation

$$(a + hm)x + (h + bm)y + (g + fm) = 0.$$

This is the equation of the diameter which bisects the chord.

This may also be shown by putting  $\frac{x_1 + x_2}{2}$  for  $x$ ,  $\frac{y_1 + y_2}{2}$  for  $y$ , the equation being satisfied thereby.

Finally: *The pole of a given line lies on the diameter which bisects the chord intercepted on that line by the conic.*

*To find the pole of a given line:* Draw the diameter conjugate to the line, join its extremities with any point of the line produced to cut the conic. Through the points of intersection draw a line to cut the diameter.

The construction in case of the parabola is easily effected by remembering that the center is at infinity.

